The Fourier Transform

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Recommended readings:

- *A Digital Signal Processing Primer*, Ken Steiglitz. Addison-Wesley, 1996.

- *The Computer Music Tutorial*, Curtis Roads. MIT Press, 1995.

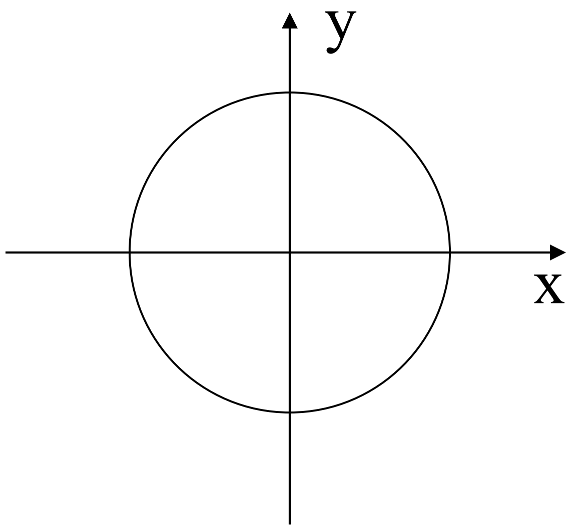
- *DSP First: A Multimedia Approach*, J. H. McClelland, R. W. Schafer, M. A. Yoder. Prentice Hall, 1998.

Basic mathematics

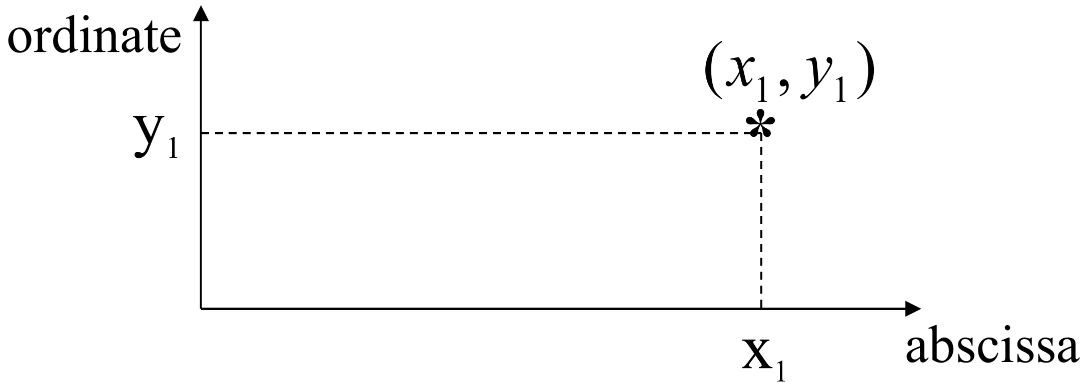
- Complex numbers

$$(x + jy) \quad \text{where } x : \text{real part}$$
$$y : \text{imaginary part}$$
$$j : \sqrt{-1}$$

- Complex plane
  x-axis (real part)
  y-axis (imaginary part)



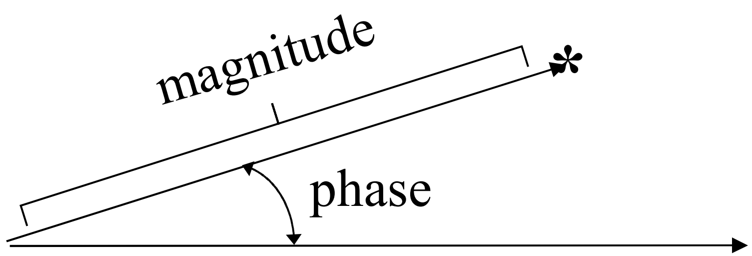
- Rectangular co-ordinates



- Polar co-ordinates

magnitude : $\sqrt{x^2 + y^2}$

phase : $tan^{-1}(y/x)$



- number e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 2.718281...$$

- Complex exponential

$$e^{(x+jy)}$$

- Sine function

$$sin(x)$$

- Euler's identity

$$e^{jx} = \cos(x) + j\,sin(x)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- linear magnitude and dB

$$\mathrm{dB} = 20\log\left(A/A_0\right)$$

Complex exponential:

$$\bar{x}(t) = Ae^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t+\phi) + jA\sin(\omega_0 t+\phi)$$

Real sinewave:

$$x(t) = A\cos(\omega_0 t+\varphi) = A\left(\frac{e^{j(\omega_0 t+\varphi)} + e^{-j(\omega_0 t+\varphi)}}{2}\right)$$

$$= \frac{1}{2}Xe^{j\omega_0 t} + \frac{1}{2}X^{*}e^{-j\omega_0 t} = \frac{1}{2}\bar{x}(t) + \frac{1}{2}\bar{x}^{*}(t)$$

$$= \mathrm{Re}\{\bar{x}(t)\}$$

Continuous Fourier transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

t : Continuous time index in seconds

ω : Continuous frequency index in radians per second

inverse transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Discrete Fourier transform (DFT)

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n}$$

$$\omega_k = 2\pi k/N,\ N \text{ even},\ k = 0,1,\ldots,N-1$$

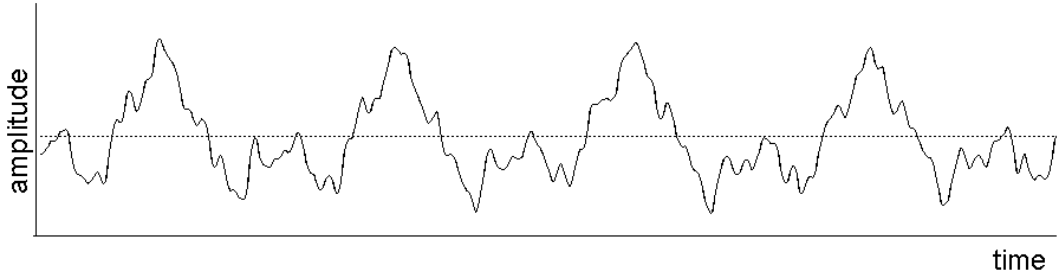
ω : discrete radian frequency,

n : discrete time index in samples,

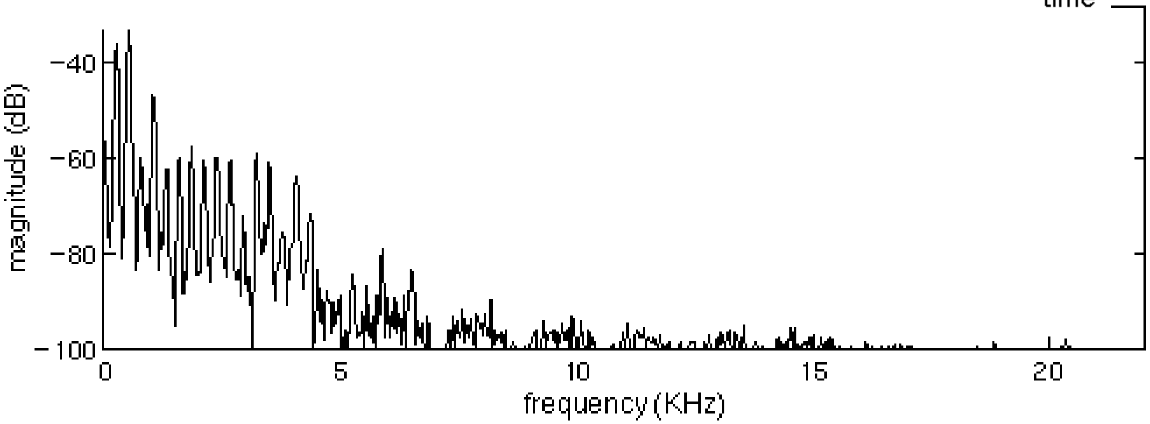
k : discrete frenquency index in bins.

Hertz-Radian relationship: $f = f_s\, \omega/2\pi$

f : frequency in Hz, f_s : sampling rate, ω : radian frequency.

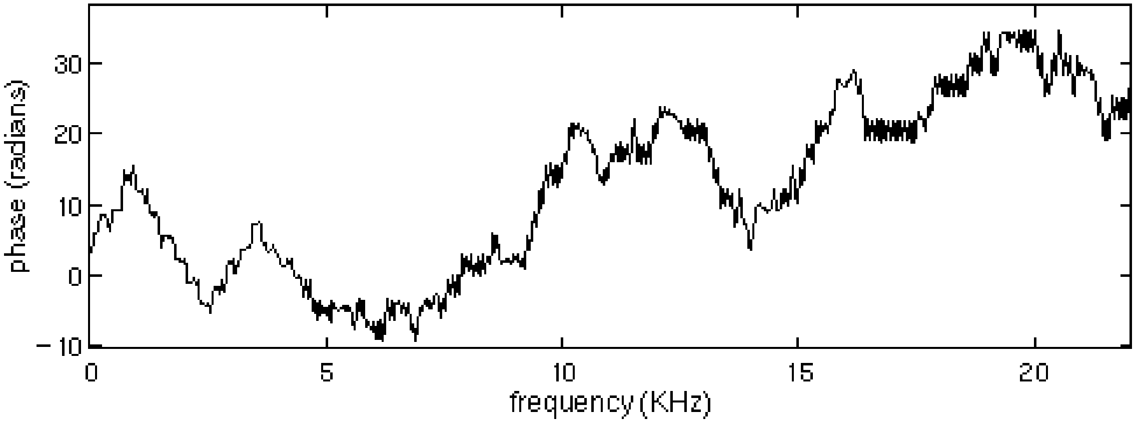


a few periods
of a piano sound
x(n)



magnitude
spectrum

$$20\log_{10}\left(\left|X(k)\right|\right)$$



phase
spectrum

$$\angle X(k)$$

Inverse DFT:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\omega_k n}$$

FFT implementation of the DFT: divide-and-conquer

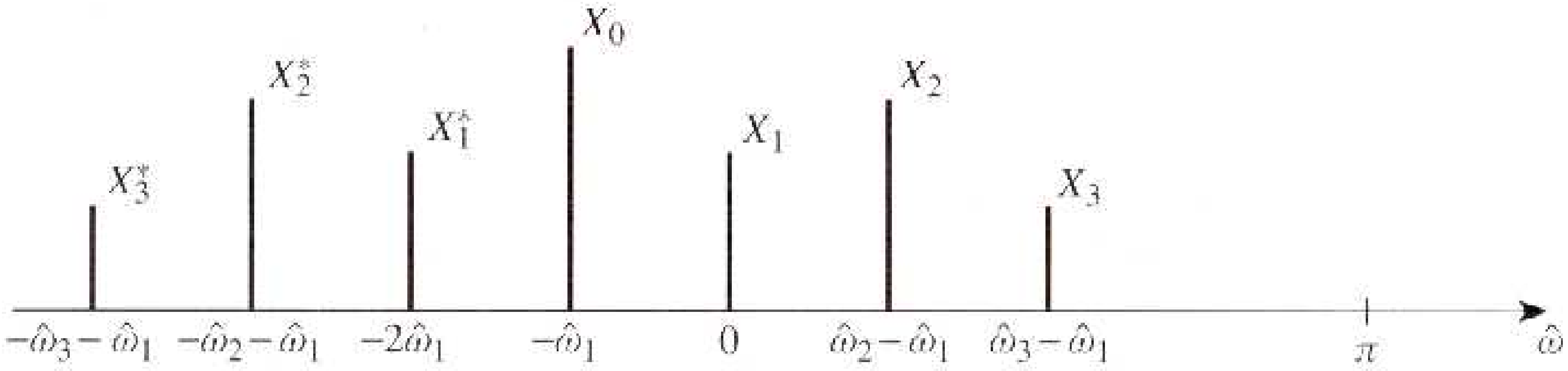
DFT: proportional to N^2
FFT: proportional to N log N

Frequency shifting

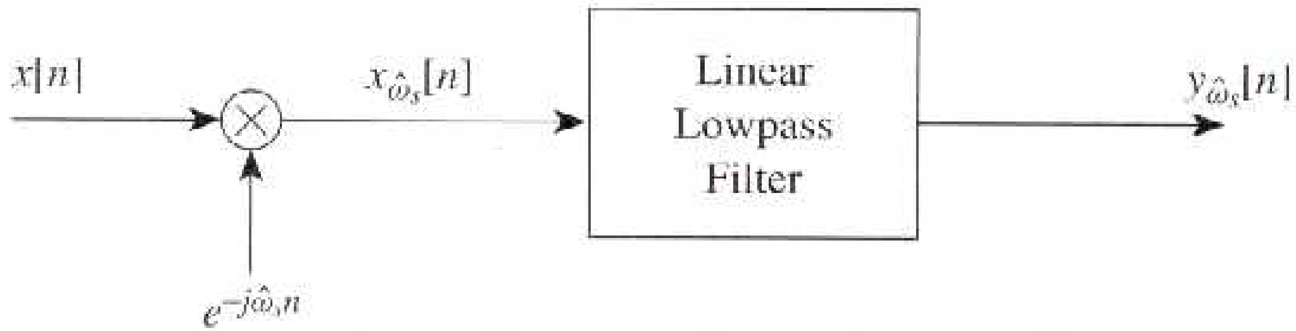
$$x_{\hat{\omega}_s}[n] = x[n] e^{-j\hat{\omega}_s n}$$

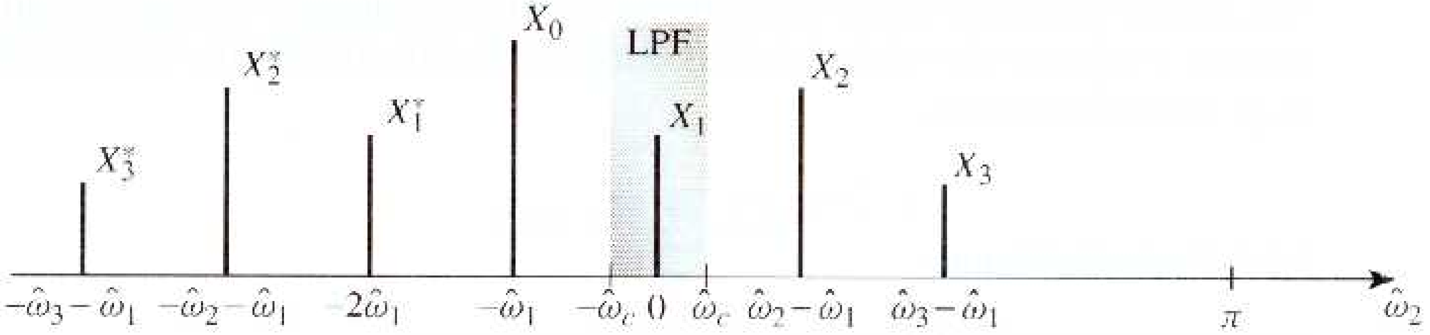
$$= \left[X_0 + \sum_{k=1}^{N} \left(X_k e^{j\hat{\omega}_k n} + X_k^* e^{-j\hat{\omega}_k n} \right) \right] e^{-j\hat{\omega}_s n}$$

$$= X_0 e^{-j\hat{\omega}_s n} + \sum_{k=1}^{N} \left(X_k e^{j(\hat{\omega}_k - \hat{\omega}_s) n} + X_k^* e^{-j(\hat{\omega}_k + \hat{\omega}_s) n} \right)$$



Channel filter





Running-sum filtering

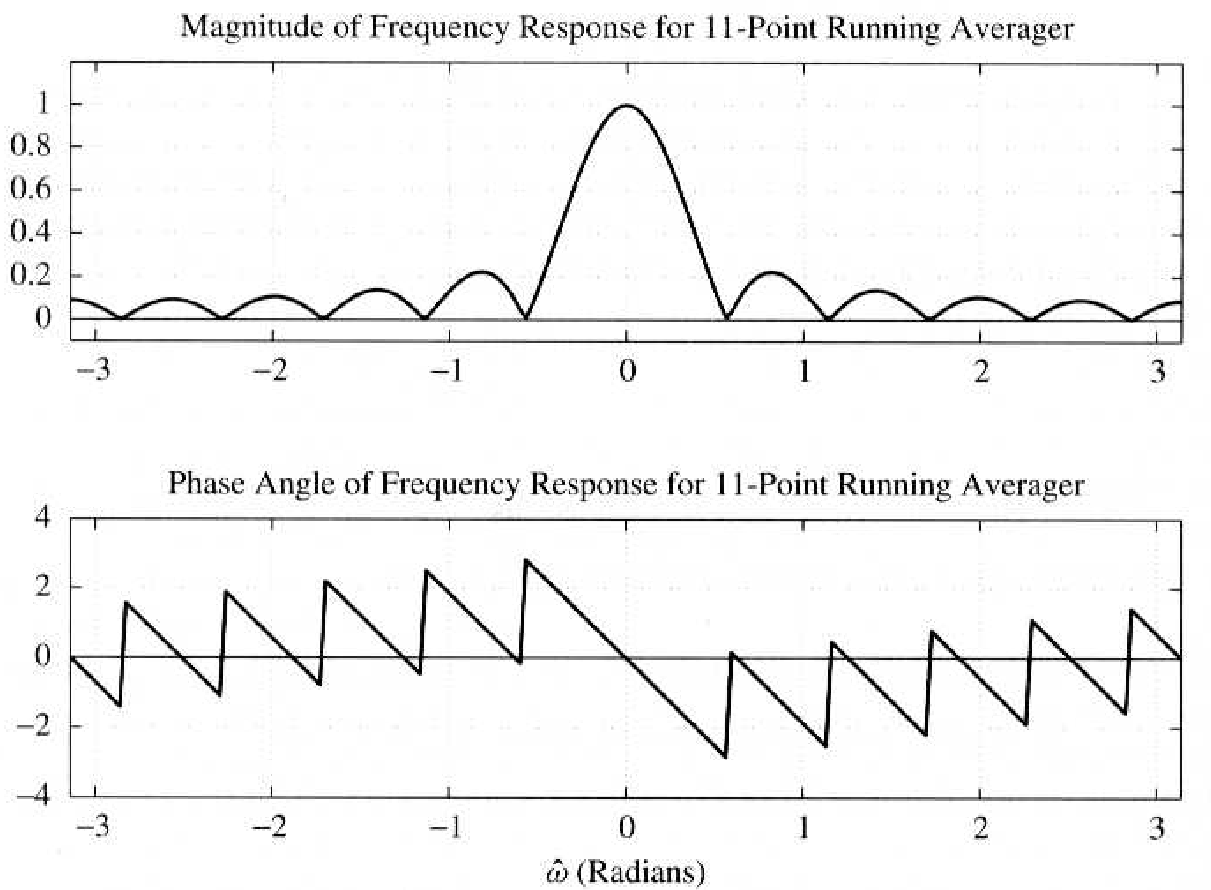
Difference equation:

$$y[n] = \sum_{l=0}^{L-1} x[n-l]$$

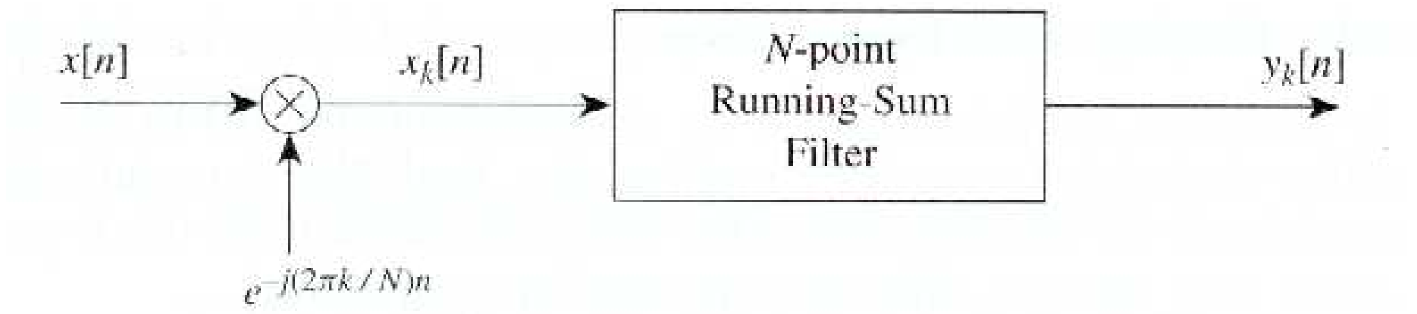
Frequency response:

$$H(\hat{\omega}) = \sum_{m=0}^{L-1} e^{-j\hat{\omega}m} = \left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)} \right) e^{-j\hat{\omega}(L-1)/2}$$

The zeros of $H(\hat{\omega})$ are equally spaced at $\hat{\omega} = 2\pi k/L$


Magnitude of Frequency Response for 11-Point Running Averager
1
0.8
0.6
0.4
0.2
0
−3
−2
−1
0
1
2
3
Phase Angle of Frequency Response for 11-Point Running Averager
4
2
0
−2
−4
−3
−2
−1
0
1
2
3
$\hat{\omega}$ (Radians)

Spectral analysis



$$x[n] = \frac{1}{N}\sum_{l=0}^{N-1} X[l] e^{j(2\pi/N)\ln}$$ periodic signal

after the multiplier:

$$x_k[n] = \frac{1}{N}\sum_{l=0}^{N-1} X[l] e^{j(2\pi/N)\ln} e^{-j(2\pi/N)kn}$$

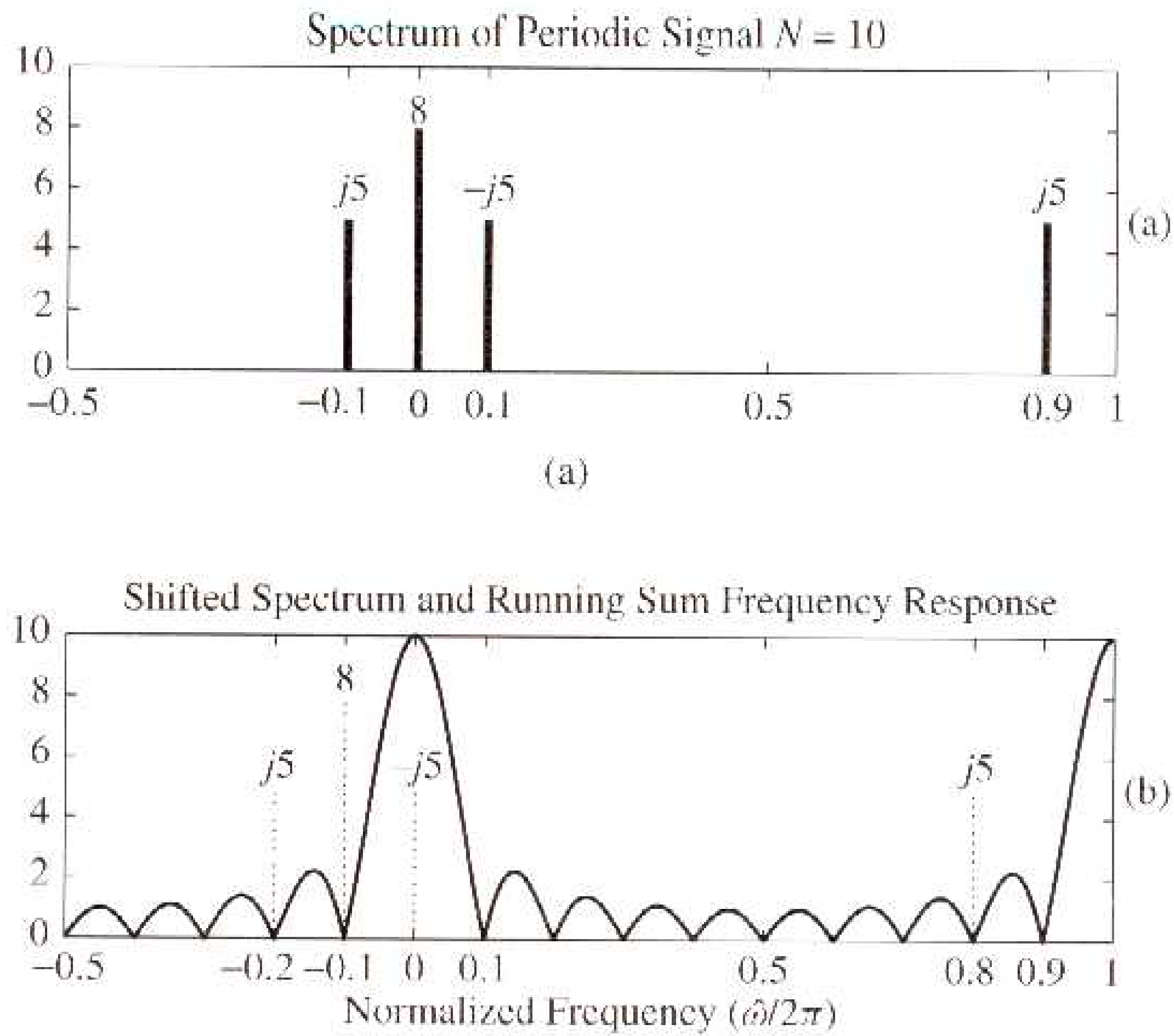
$$y_k[n] = \frac{1}{N}\sum_{l=0}^{N-1} H\left(e^{j2\pi(l-k)/N}\right)X[l]e^{j(2\pi/N)(l-k)n}$$

$$= \frac{1}{N}H\left(e^{j0}\right)X[k]$$

$$+\frac{1}{N}\sum_{\substack{l=0\\l\neq k}}^{N-1} H\left(e^{j2\pi(l-k)/N}\right)X[l]e^{j(2\pi/N)(l-k)n}$$

since $H\left(e^{j0}\right)=N$ and $H\left(e^{j2\pi(l-k)/N}\right)=0$ when $(l-k)\neq 0$

$$y_k[n] = X[k]$$



Spectrum of Periodic Signal N = 10
10
8
6
4
2
0
8
j5
−j5
j5
(a)
−0.5
−0.1
0
0.1
0.5
0.9
1
Shifted Spectrum and Running Sum Frequency Response
10
8
6
4
2
0
8
j5
−j5
j5
(b)
−0.5
−0.2
−0.1
0
0.1
0.5
0.8
0.9
1
Normalized Frequency (ω̂/2π)

(a)

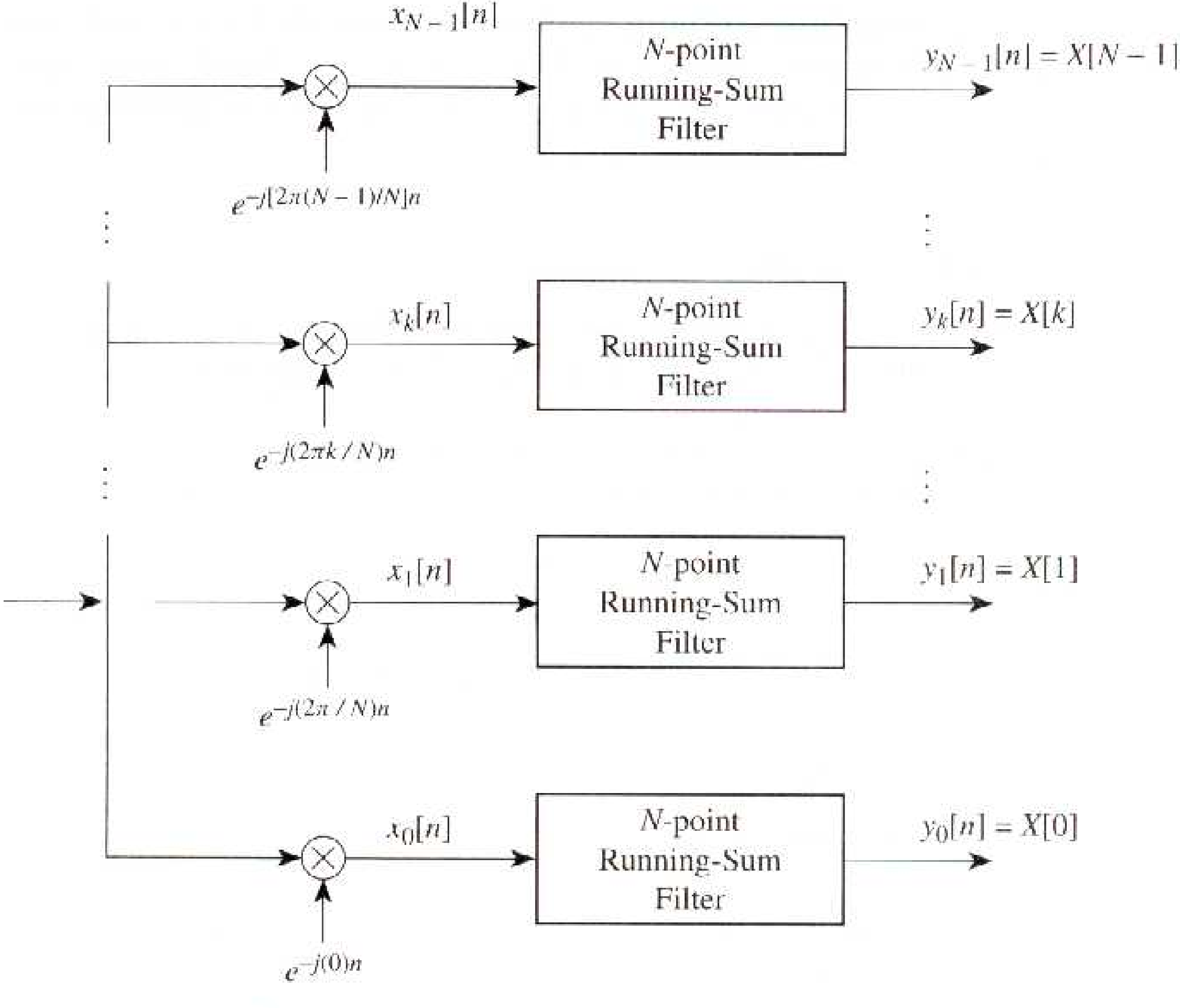
(b)

$$y_k[n] = \sum_{l=0}^{N-1} x_k[n-l]$$

$$= \sum_{m=n-N+1}^{n} x_k[m] = \sum_{m=n-N+1}^{n} x[m] e^{-j(2\pi/N)km}$$

using $n = N-1$

$$X[k] = \sum_{m=0}^{N-1} x[m] e^{-j(2\pi/N)km} \qquad k = 0,1,2,\ldots,N-1$$



$x_{N-1}[n]$
N-point Running-Sum Filter
$y_{N-1}[n] = X[N-1]$
$e^{-j[2\pi(N-1)/N]n}$
$x_k[n]$
N-point Running-Sum Filter
$y_k[n] = X[k]$
$e^{-j(2\pi k/N)n}$
$x_1[n]$
N-point Running-Sum Filter
$y_1[n] = X[1]$
$e^{-j(2\pi/N)n}$
$x_0[n]$
N-point Running-Sum Filter
$y_0[n] = X[0]$
$e^{-j(0)n}$

the DFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} \qquad n = 0,1,2,\ldots,N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} \qquad k = 0,1,2,\ldots,N-1$$

DFT properties

$$x \leftrightarrow X \text{ (transform pairs)}$$

$$X = DFT(x), x = IDFT(X)$$

- **Linearity:**

  $ax_1 + bx_2 \leftrightarrow aX_1 + bX_2$ (mixing commutes with the DFT)

- **Convolution:**

  convolution $\leftrightarrow$ point - by - point multiplication

- **Shift:**

  shift $\leftrightarrow$ multiplication by a complex exponential

- **Evenness:**

  even $\leftrightarrow$ real - valued

  (even function : for every k, $x_k = x_{-k}$)

- **Zero padding:**

  zero padding $\leftrightarrow$ interpolation

- **Power:**

$$\sum_{n=0}^{N-1} \left| x(n) \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| X(k) \right|^2 \qquad \text{(Rayleigh)}$$

DFT examples

$$x_1[n] = e^{j(2\pi k_0/N)n} \quad \text{for} \quad n = 0,1,2,\ldots,N-1$$

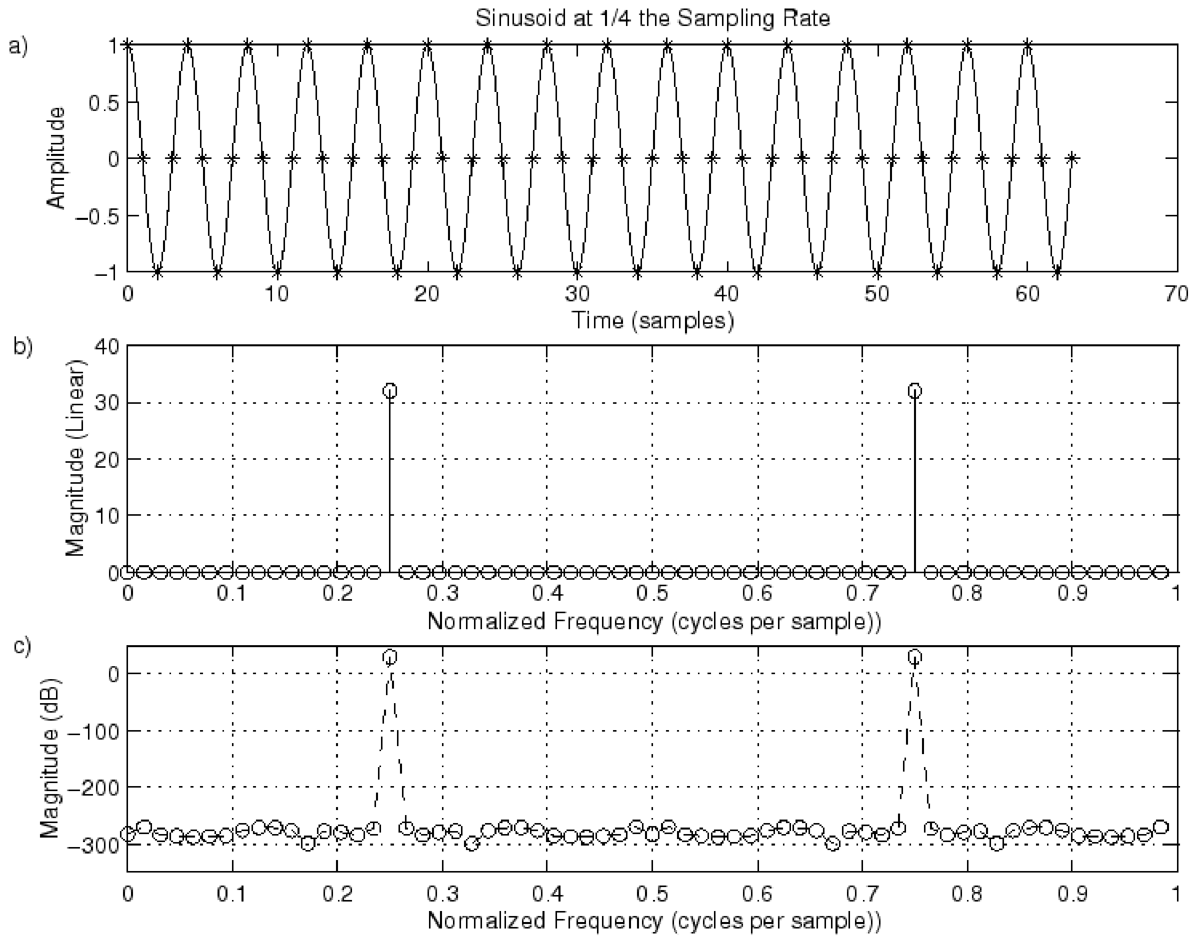
$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{N-1} e^{j(2\pi k_0/N)n} e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{N-1} e^{-j(2\pi/N)(k-k_0)n}$$

$$= 1 + e^{-j(2\pi/N)(k-k_0)} + e^{-j(2\pi/N)(k-k_0)2} + \ldots + e^{-j(2\pi/N)(k-k_0)(N-1)}$$

$$= \frac{1 - e^{-j(2\pi/N)(k-k_0)N}}{1 - e^{-j(2\pi/N)(k-k_0)}} = N\delta[k - k_0]$$


Sinusoid at 1/4 the Sampling Rate
a)
Amplitude
1
0.5
0
−0.5
−1
0
10
20
30
40
50
60
70
Time (samples)
b)
Magnitude (Linear)
40
30
20
10
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (cycles per sample))
c)
Magnitude (dB)
0
−100
−200
−300
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency (cycles per sample))

$$x_3[n] = e^{j(\hat{\omega}_0 n+\phi)} \quad \text{for} \quad n = 0,1,2,\ldots,N-1$$

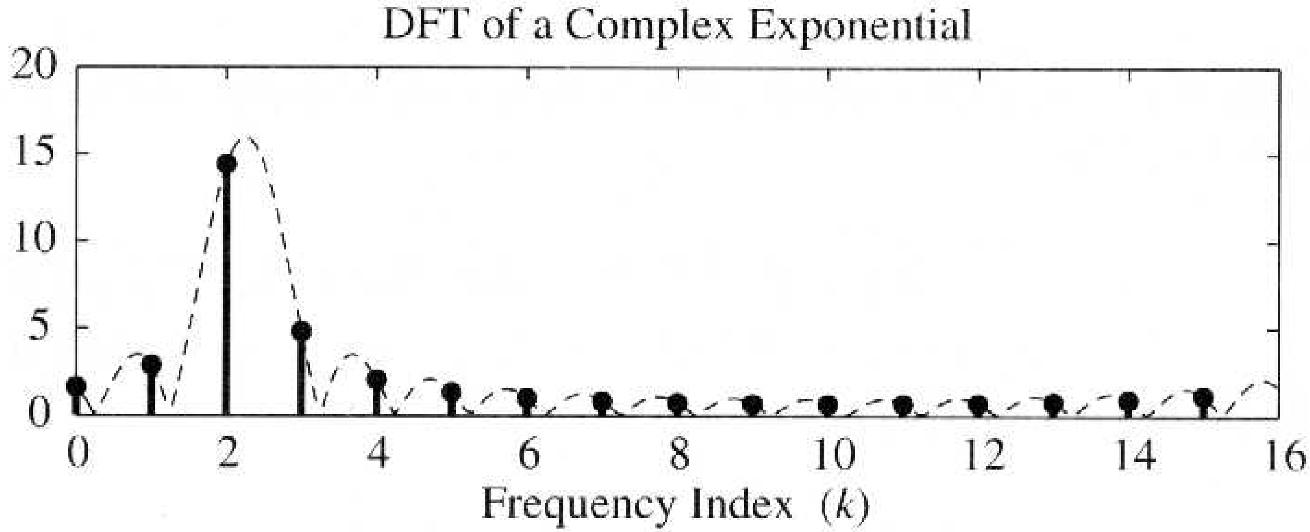
$$X_3[k] = \sum_{n=0}^{N-1} e^{j(\hat{\omega}_0 n+\phi)} e^{-j(2\pi/N)kn}$$

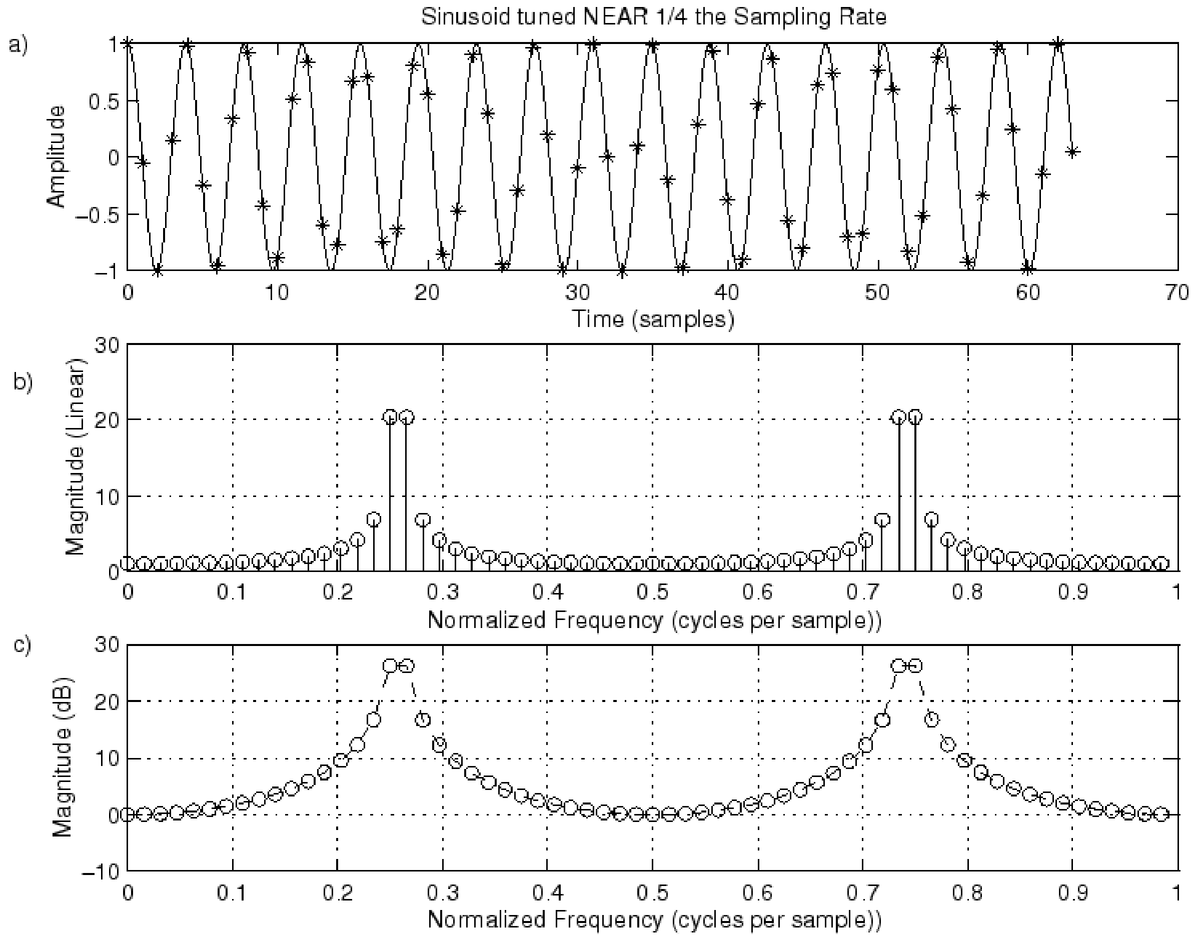
$$= e^{j\phi} \sum_{n=0}^{N-1} e^{-j(2\pi k/N-\hat{\omega}_0)n}$$

$$= e^{j\phi} \left(e^{-j(0)} + e^{-j(2\pi k/N-\hat{\omega}_0)} + \ldots + e^{-j(2\pi k/N-\hat{\omega}_0)(N-1)} \right)$$

$$= e^{j\phi} \frac{1-e^{-j(2\pi k/N-\hat{\omega}_0)N}}{1-e^{-j(2\pi k/N-\hat{\omega}_0)}}$$

$$= e^{j\phi} e^{-j(2\pi k/N-\hat{\omega}_0)(N-1)/2} \frac{\sin\left(\left(2\pi k/N-\hat{\omega}_0\right)N/2\right)}{\sin\left(\left(2\pi k/N-\hat{\omega}_0\right)/2\right)}$$


DFT of a Complex Exponential
20
15
10
5
0
0
2
4
6
8
10
12
14
16
Frequency Index (k)


Sinusoid tuned NEAR 1/4 the Sampling Rate
a)
Amplitude
Time (samples)
b)
Magnitude (Linear)
Normalized Frequency (cycles per sample))
c)
Magnitude (dB)
Normalized Frequency (cycles per sample))