Variations on the Gaussian Multiple-Access Channel

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Abstract—This paper studies two models derived from the Gaussian Multiple-Access Channel. In each, information is sent by energy modulation, rather than on the complex signal amplitude. For continuous energy, the model is the additive exponential noise (AEN) channel, whose capacity region equals that of the equivalent Gaussian channel at identical noise and energy levels. Differently from the Gaussian case, its capacity region is not equivalent Gaussian channel at identical noise and energy levels. An exception is the AEN channel, whose capacity region equals that of the AWGN channel, whose capacity region equals that of the AWGN. An outer bound to its capacity region is derived, and shown to be strictly contained in the region for the additive exponential/Gaussian noise channel.

I. The Additive Energy Channel: A Variation on Additive Gaussian Noise

In this paper, we study several variations of the discrete-time multiple-access Gaussian channel. In all cases, the output variable at instant $k$, denoted by $Z_k$, has the form

$$Z_k = X_{k,1}(s_{k,1}) + X_{k,2}(s_{k,2}) + W_k,$$  \hspace{1cm} (1)

namely a sum of variables $X_{k,i}$, which are a function (possibly stochastic) of the corresponding input $s_{k,i}$, and a noise $W_k$.

For the sake of simplicity, we consider two users, $i = 1, 2$, since generalization to $m$ users poses no conceptual problems.

In the classical complex-valued additive white Gaussian noise (AWGN) channel the channel inputs $s_{k,i}$ and output $z_k$ are complex numbers. Next to it, we consider

1) The additive exponential noise channel (AEN).
2) The additive energy channel (AE-Q).

The alternative models we study are non-coherent, in the sense that information is sent by modulating a signal energy $|s_{k,i}|^2$, rather than the complex amplitude $s_{k,i}$ itself.

One motivation for this work is to study how sensitive information-theoretic results are to changes on the underlying channel model. For example, phase noise, or in general the cost incurred by carrier synchronization – especially in multiuser environments–, are somewhat difficult to include in an AWGN model. A non-coherent alternative, such as the additive energy model, might turn out to be more accurate under some circumstances.

The first variation is the additive exponential noise (AEN) channel. The AEN channel, including a multiple-access scenario, was studied by Verdú [1]. We establish a correspondence with the AWGN channel by setting the input

$$s_{k,i}^{\text{AEN}} = |s_{k,i}^{\text{AWGN}}|^2.$$  \hspace{1cm} (2)

For the AEN model, $Z_k, S_{k,i}, W_k$ are real-valued and positive. Further, we have $X_{k,i} = S_{k,i}$, for both AWGN and AEN channels. We will enforce $E[|S_{k,i}^{\text{AWGN}}|^2] = E[S_{k,i}^{\text{AEN}}] = E_s$.

In AWGN, the noise component $W_k$ is drawn according to a circularly-symmetric complex Gaussian density, with variance $\sigma_w^2$ [2]. The AEN noise $W_k$ is obtained from the AWGN value, $w_k^{\text{AEN}} = |w_k^{\text{AWGN}}|^2$, and has thus an exponential density [2], of mean $E_w = \sigma_w^2$.

Channels are time-invariant and memoryless, with output conditional density $Q(z|s_1, s_2)$

$$Q(z|s_1, s_2) = \frac{1}{\pi \sigma_w^2} e^{-\frac{|z - s_1 - s_2|^2}{\sigma_w^2}}, \text{ for AWGN}$$  \hspace{1cm} (4)

$$Q(z|s_1, s_2) = \frac{1}{E_w} e^{-\frac{|z - s_1 - s_2|^2}{E_w}}, \text{ for AEN},$$  \hspace{1cm} (5)

where $u(t)$ is a step function.

We hasten to remark that $z_k^{\text{AEN}}$ is not related to $z_k^{\text{AWGN}}$ by $z_k^{\text{AEN}} = |z_k^{\text{AWGN}}|^2$. Indeed, in a single-user transmission,

$$|z_k^{\text{AWGN}}|^2 = x_k^{\text{AEN}} + w_k^{\text{AEN}} + 2 \text{Re}(x_k^{\text{AWGN}} w_k^{\text{AWGN}}^*),$$  \hspace{1cm} (6)

and $z_k^{\text{AEN}} \neq |z_k^{\text{AWGN}}|^2$ in general. Implicit in the construction of the AEN channel lies an orthogonality property between signal $x_k$ and noise $w_k$, such that $x_k^{\text{AEN}}, w_k^{\text{AEN}} = 0$ in some (difficult to make precise) sense. By analogy, we postulate a generalization of this condition to orthogonality among users, such that $x_{k,1}, x_{k,2}^* = 0$, for some ( unspecified) cross-product.

The AEN channel has received little attention, even though its simplicity matches that of the AWGN. An exception is Verdú’s work [1]. The non-coherent AWGN channel, with an output of the form in Eq. (6), has also been considered in the literature [3], and references therein.

The second variation is the quantized additive energy (AE-Q) channel. We assume the existence of a quantum of energy, such that the channel output comes in a discrete number of quanta. The variables $Z_k, X_{k,i}$ and $W_k$ in Eq. (1) are now positive integers. As example, we identify the quanta with photons, and give the energy of a photon as $h\nu$, where $h$ is Planck’s constant and $\nu$ is its frequency.

1When comparing channels, we may use a superscript to indicate the channel which the variable belong to.
The input is a positive real number $s_k \geq 0$. We enforce
\begin{equation}
\rho_{k,i}^{AE-Q} = \rho_{k,i}^{AEN} = |\rho_{k,i}^{AWGN}|^2.
\end{equation}
We postulate that the output $X_{k,i}$ is distributed as
\begin{equation}
\Pr(X_{k,i} = m|S_{k,i} = s) = e^{-s} \frac{s^m}{m!},
\end{equation}
that is, a Poisson distribution with parameter $s_{k,i}$. This assumption leads to a fundamental difference with previous models, as noise is now (partly) signal-dependent.

We assume that $W_k$ has a geometric distribution,
\begin{equation}
\Pr(W_k = m) = \frac{1}{1 + \epsilon_w} \left( \frac{\epsilon_w}{1 + \epsilon_w} \right)^m,
\end{equation}
where $\epsilon_w$ is the mean, given by $\epsilon_w \nu = \sigma^2 = E_w$, the respective values in the AWGN and AEN channels. Table I shows the value of $\epsilon_w$ for several frequencies at $T_0 = 290$ K, computed by using the blackbody radiation formula [4].

| TABLE I |
| TYPICAL VALUES OF $\epsilon_w$ AT $T_0 = 290$ K. |
| Frequency | 6 MHz | 6 GHz | 60 GHz | 4 THz | 500 THz |
| Wavelength | 50 m | 5 cm | 0.5 cm | 75 cm | 600 nm |
| $\epsilon_w$ | $10^6$ | 1000 | 100 | 1 | $10^{-36}$ |

For fixed input values $s_1$ and $s_2$, the output variable $Z$ is described by a probability distribution, which can be expressed as a convolution of Poisson and geometric distributions, in an analogous manner to Eqs. (4) and (5).

To the best of our knowledge, this channel model is new. Some results on the capacity of the single-user case were presented in [5], under the name “Einstein radiation channel”. A variation, when $w_k = 0$ and therefore $w_k = 0$, has been studied under the name discrete-time Poisson channel. References are [6]–[8].

Each user is subject to an energy constraint, whose form depends on the channel. For AWGN, we have $\sum_{k=1}^n |s_{k,i}|^2 \leq n E_{s,i}^{AWGN}$. For AEN, the natural constraint is $\sum_{k=1}^n s_{k,i} \leq n E_{s,i}^{AEN}$. And finally, for the AE-Q channel, the constraint is $\sum_{k=1}^n s_{k,i} \leq n E_{s,i}^{AE-Q}$. We relate $\epsilon_s$ with the constraint on the signal energy in the AWGN and AEN channels, by
\begin{equation}
\epsilon_s \nu = E_{s,i}^{AWGN} = E_{s,i}^{AEN}.
\end{equation}

II. BOUNDS TO THE SINGLE-USER CAPACITY

We first review some results on the single-user versions of the channels described in Sect. I. The usual definitions of code, encoding and decoding operations, and channel capacity for the AWGN channel apply here for the appropriate alphabets and channel constraints [2], [9].

The capacity $C$ (in nats per channel use) of the AWGN channel [2], [9] and of the equivalent AEN channel [1] is
\begin{equation}
C = \log(1 + \gamma),
\end{equation}
where $\gamma$ is the signal-to-noise ratio, $\gamma = E_s/\sigma^2$ for AWGN and $\gamma = E_s/E_w$ for the AEN channel. Note that the capacity does not depend on the absolute energy levels, but on their ratio. This will not be the case for the AE-Q channel.

In the AEN model, the capacity-achieving input has density
\begin{equation}
p_Z(s) = \frac{E_s}{(E_s + E_w)^2} e^{-\frac{s^2}{E_s + E_w}} + \frac{E_w}{E_s + E_w} \delta(s),
\end{equation}
for $s \geq 0$ [1]. For this input, the output density $p_Z(z)$ has exponential density with mean $E_s + E_w$.

For the AE-Q channel, we resort to providing tight upper and lower bounds to its capacity $C(\epsilon_s, \epsilon_w)$.

**Theorem 1** The capacity $C(\epsilon_s, \epsilon_w)$ of the Additive Energy-Quantized channel is upper bounded by
\begin{equation}
C^+(\epsilon_s, \epsilon_w) \leq \min(C_L, C_H),
\end{equation}
where $C_L$ and $C_H$ are respectively given by
\begin{equation}
C_L = H_{\text{Geom}}(\epsilon_s + \epsilon_w) - H_{\text{Geom}}(\epsilon_w)
\end{equation}
and
\begin{equation}
C_H = \log \left( \frac{1 + \sqrt{2} \epsilon_s + \frac{1}{\sqrt{1 + 2 \epsilon_s}}}{1 + 2 \epsilon_s} \right).
\end{equation}
Here $H_{\text{Geom}}(\alpha)$ is the entropy of a geometric distribution with mean $\alpha$.

\begin{equation}
H_{\text{Geom}}(\alpha) = (1 + \alpha) \log(1 + \alpha) - \alpha \log \alpha.
\end{equation}
Note that $C_H$ does not depend on $\epsilon_w$, as it is due to the Poisson noise. In addition, $C_H$ behaves as $\frac{1}{2} \log \epsilon_s$ for $\epsilon_s \to \infty$.

**Proof:** For any input, we have
\begin{equation}
I(S; Z) = H(Z) - H(Z|S)
\end{equation}
\begin{equation}
\leq H_{\text{Geom}}(\epsilon_s + \epsilon_w) - H(X(S) + W|S),
\end{equation}
as the geometric distribution has the highest entropy under the given constraints [9]. Then,
\begin{equation}
H(X(S) + W|S) \geq H(W|S) = H(W),
\end{equation}
because the entropy of a sum of two independent random variables is larger than the entropy of each of them (Exercise 18 of Chapter 2 of [9]), and $W$ and $S$ are independent. Therefore,
\begin{equation}
I(S; Z) \leq H_{\text{Geom}}(\epsilon_s + \epsilon_w) - H(W)
\end{equation}
\begin{equation}
= H_{\text{Geom}}(\epsilon_s + \epsilon_w) - H_{\text{Geom}}(\epsilon_w).
\end{equation}
As this holds for all inputs, this proves the formula for $C_L$.

Then, we note that the variables $S, X(S)$, and $Z(X)$ form a Markov chain, $S \rightarrow X(S) \rightarrow Z = X(S) + W$. Then, by the data processing inequality,
\begin{equation}
I(S; Z) \leq I(S; X(S)),
\end{equation}
that is, the mutual information achievable in the discrete-time Poisson (DT-P) channel. In [10], the expression in Eq. (15) was computed as upper bound to the capacity of the DT-P channel. 


Theorem 2 A lower bound to the capacity $C(\epsilon_s, \epsilon_w)$ of the Additive Energy-Quantized channel is

$$C^{-}(\epsilon_s, \epsilon_w) \triangleq H_{\text{Geom}}(\epsilon_s + \epsilon_w) - \frac{\epsilon_w}{\epsilon_s + \epsilon_w} H_{\text{Geom}}(\epsilon_w) - \frac{\epsilon_s}{(\epsilon_s + \epsilon_w)^2} \int_{0}^{\infty} H_{\text{max}}(Z|s) e^{-\frac{s}{\epsilon_s + \epsilon_w}} ds,$$

where

$$H_{\text{max}}(Z|s) \triangleq \frac{1}{2} \log 2\pi e \left( \text{Var}(Z|s) + \frac{1}{12} \right).$$

Proof: It is possible to show (see [5]) that Eq. (12), with $E_s = \epsilon_s$, and $E_w = \epsilon_w$, when used in the AE-Q channel, induces a geometric distribution at the output with parameter $\epsilon_s + \epsilon_w$. The output entropy is therefore

$$H(Z) = H_{\text{Geom}}(\epsilon_s + \epsilon_w).$$

As for the conditional entropy,

$$H(Z|S) = \int_{0}^{\infty} H(Z|s) p_S(s) ds,$$

we consider two cases. When $s = 0$ the output is geometric of parameter $\epsilon_w$. For $s > 0$, the entropy $H(Z|S = s)$, we use the following upper bound for the entropy of an integer-valued random variable with a fixed variance [9]

$$H(Z|s) \leq \frac{1}{2} \log 2\pi e \left( \text{Var}(Z|s) + \frac{1}{12} \right).$$

Using the definition of the mutual information $I(S;Z) = H(Z) - H(Z|S)$, the theorem follows.

Fig. 1 shows as a function of the input number of photons, $\epsilon_s$, and for several values of $\epsilon_w$, the upper and lower bounds to the capacity in Theorems 1 and 2. For low values of $\epsilon_s, C_L$ is tighter, whereas for higher values of $\epsilon_s, C_H$ gives the best bound. The transition approximately occurs when $C_L = C_H$. For large enough photon counts, this is equivalent to $\epsilon_s = \epsilon_w(1 + \epsilon_w) \simeq \epsilon_w^2$. If we define the signal-to-noise ratio as $\gamma = \epsilon_s/\epsilon_w$, which coincides with the definition for the AWGN and AEN channels, the crossing between $C_L$ and $C_H$ takes place at approximately $\gamma^* = \epsilon_w$, about 20 dB at 60 GHz and 30 dB at 6 GHz, increasing by 10 dB per decade.

An interesting corollary, which we do not prove in detail (it is an exercise in calculus) relates the capacity of the AE-Q channel with that of the AEN and AWGN cases.

Corollary 1 For finite values of $\epsilon_s$ and $\epsilon_w$, the capacity of the Additive Energy-Quantized channel is strictly upper bounded by the capacity of the additive exponential noise channel,

$$C(\epsilon_s, \epsilon_w) < \log(1 + \gamma).$$

However, as $\epsilon_w \to \infty$, the upper bound essentially coincides with the capacity of the additive exponential noise channel,

$$C_L = \log(1 + \gamma) + O(\epsilon_w^{-1}).$$

In fact, not only is $C_L \simeq \log(1 + \gamma)$, but Fig. 1 suggests that the rate may be achievable, especially if we did not use the approximation (27) in Theorem 2.

III. TWO-USER MULTIPLE ACCESS CHANNEL

As it happened in the single-user case, the conventional definitions for code, encoding and decoding operations, as well as for the achievable rates and capacity region for the AWGN channel apply here [9]. Hence, for the two-user multiple access channel, the capacity region of $R$ is the closure of the convex hull of all rate pairs $(R_1, R_2)$ satisfying

$$0 \leq R_1 \leq I(S_1;Z|S_2)$$

$$0 \leq R_2 \leq I(S_2;Z|S_1)$$

$$R_1 + R_2 \leq I(S_1,S_2;Z),$$

for all probability assignments $p_{S_1,S_2}($) = p_{S_1}($)p_{S_2}($)$. For the complex-valued AWGN [2], [9], as well as for the AEN [1], the capacity region is the pentagon

$$R = \{(R_1, R_2): 0 \leq R_1 \leq \log(1 + \gamma_1) \leq 0 \leq R_2 \leq \log(1 + \gamma_2) \leq R_1 + R_2 \leq \log(1 + \gamma_1 + \gamma_2)\}.$$
where $C^+(\epsilon_s, \epsilon_w)$ is as in Theorem 1.

**Proof:** In the AEN channel, for which other users do not contribute with noise when their input is known, one has

$$I(S_1; Z|S_2) = I(S_1; X_1 + W).$$  \hspace{1cm} (39)

In the AE-Q channel, the residual Poisson noise from $S_2$ remains, and this equation appears as an inequality. This is done by noting that applying the data processing inequality to the Markov chain

$$S_1 \rightarrow X(S_1) \rightarrow X_1 + W \rightarrow X_1 + X_2 + W.$$  \hspace{1cm} (40)

Hence,

$$I(S_1; Z|S_2) = I(S_1; X_1 + X_2 + W|S_2) \leq I(S_1; X_1 + W|S_2) = I(S_1; X_1 + W),$$  \hspace{1cm} (41-43)

as $S_2$ is independent from $X_1 + W$ and $S_2$. This value is the mutual information between a user 1 and the receiver in absence of user 2, namely the single-user scenario upper bound in Theorem 1. Reversing the roles of users 1 and 2 gives

$$I(S_2; Z|S_1) \leq I(S_2; X_2 + W) \leq C^+(\epsilon_s, \epsilon_w).$$  \hspace{1cm} (44)

Finally, Eq. (38) upper bounds the aggregate rate by that of a unique user with total energy $\epsilon_1 + \epsilon_2$.

The following result follows from Corollary 1 to Theorem 1.

**Corollary 2** For finite values of $\epsilon_s$ and $\epsilon_w$, the capacity region of the additive energy channel $\mathcal{R}^{AE-Q}$ is strictly smaller than the capacity of the additive exponential noise channel,

$$\mathcal{R}^{AE-Q} \subset \mathcal{R}^{AEN} = \mathcal{R}^{AWGN}. \hspace{1cm} (45)$$

Regarding the achievability, the AWGN/AEN approach can be followed, using superposition, successive decoding, and time-sharing. However, we have obtained only inner bounds, in line with Theorem 2. When Poisson noise is negligible, that is $\epsilon_w \rightarrow \infty$ for fixed $\epsilon_s$, one expects the AE-Q channel to get ever closer to the AEN model and, then, the shrinking of the capacity region to be limited. Due to space limitations, we do not discuss it further, and concentrate on the outer bound.

Theorem 3 gives, together with the asymptotic limit of $C_H$,

**Corollary 3** For a fixed value of $\epsilon_w$, and as the number of users $u$ becomes large, the sum rate $\sum_{i=1}^u R_i^{AE-Q}$ grows as

$$\lim_{u \to \infty} \frac{\sum_{i=1}^u R_i^{AE-Q}}{\log \sum_{i=1}^u \epsilon_s} \leq \frac{C_H(\sum_{i=1}^u \epsilon_s, \epsilon_w)}{\log \sum_{i=1}^u \epsilon_s} = \frac{1}{2}. \hspace{1cm} (46)$$

This asymptotic result should be compared with the classical

$$\lim_{m \to \infty} \frac{\sum_{i=1}^m R_i^{AEN}}{\log \sum_{i=1}^m E_{s,i}} = 1. \hspace{1cm} (47)$$

The rule of thumb separating the L and H regimes for the single-user case, $\gamma^* = \epsilon_w$, can be modified to include the number of users $u$. The threshold signal-to-noise ratio becomes

$$u \epsilon_s \simeq \epsilon_w^2 \implies \gamma^* = \frac{\epsilon_w}{u}. \hspace{1cm} (48)$$

Assuming say, 100 users, the threshold $\gamma$ is 20 dB lower than the numerical values used in the single-user scenario, that is about 0 dB at 60 GHz and 10 dB at 6 GHz.

Fig. 2(a), shows upper bounds to the achievable rates when $\epsilon_s = 10^5$ photons and $\epsilon_w = 5 \cdot 10^4$ photons, for several values of $\epsilon_w$, corresponding to different frequencies, 0.6, 6, and 60 GHz. The Poisson noise limit $C_H$ is constant, and shown in solid lines; the thermal limit changes, and exceeds the shot limit for sufficiently high frequencies. Fig. 2(b), shows results for fixed $\epsilon_w = 3000$ photons, or about 2 GHz and different signal-to-noise ratios; both users have the same signal energy, that is $\epsilon_s = \gamma \epsilon_w$. Again, the thermal noise limit prevails for low $\gamma$, and at high enough $\gamma$ the shot limit takes over.
IV. Effect of Feed-back on the Capacity Region

Feedback from the receiver to the source does not increase the capacity of the single-user discrete memoryless channel, and neither does that of the single-user AWGN channel [9]. In the multiple-access channel, feedback from the receiver to the sources may however enlarge the capacity region [9]. In Ozarow’s scheme which achieves all points in the capacity region, and a converse. In Ozarow’s scheme, the inputs $S_1$ and $S_2$ are correlated, with a correlation coefficient $\rho = \frac{E[S_1 S_2]}{\sqrt{E[S_1^2] E[S_2^2]}}$. An essential observation is that the total received energy $E_{\text{rec}}$ is

$$E_{\text{rec}} = E[(|S_{c,1} + S_{c,2}|^2)] = E_s + E_{s,2} + 2\text{Re}(\rho)\sqrt{E_s E_{s,2}},$$

which can, for a judicious choice of $\rho$, exceed the transmitted energy. In some sense, the channel creates energy. Eq. (50) leads to a natural bound on $\text{Var} Z$ and $H(Z)$,

$$H(Z) \leq \log(\pi e (E_{\text{rec}} + \sigma^2)).$$

In the converse, this equation is key to prove the capacity region enlargement. Further details can be found in Ozarow’s paper [11]. The main result we need is that

$$R_{\text{AWGN}} \subseteq R_{\text{fb}}.$$

Now, for the AEN and AE-Q channel models, there is no way of increasing the received energy above the sum of the transmitted values, in the sense that

$$E[Z] = E_s + E_{s,2} + E_w.$$

This simple observation leads to the surprising

**Theorem 4** Feed-back does not increase the capacity of the additive exponential noise channel, that is

$$R_{\text{AEN}} = R_{\text{fb}}.$$

An analogous result holds for the AE-Q channel. This should be compared with the AWGN case, see Eq. (52) above.

Proof: The converse part of the coding theorem, where $H(Z), H(Z|S_2), H(Z|S_1)$ are bounded by using the respective constraints on the total energy, remains unchanged in presence of feedback. The outer region is thus unchanged.

V. Discussion

In this paper, we have studied two variations on the Gaussian Multiple-Access Channel: the additive exponential noise (AEN) channel and the quantized additive energy channel (AE-Q). In each of them, information is sent by energy modulation, rather than on the complex amplitude of the input signals. For an AWGN channel of given noise and signal levels, equivalent additive energy models have been derived.

In the single-user channel, the AEN capacity coincides with the AWGN capacity. For fixed frequency, the AE-Q capacity is close to the AWGN below a signal-to-noise ratio threshold; above the capacity-cost function changes from $\log e_s$ to $\frac{1}{2} \log e_s$. This threshold is 30 dB at 6 GHz, and increases by 10 dB every time frequency is lowered by a decade.

The fundamental assumption is the additivity of signal energy and noise energy. This assumption, however simple it may seem, differs from some common practice. In the context of single-user non-coherent Gaussian channel (see e.g. [3]), a cross-term between signal and noise appears, which has a negative effect on the capacity. On the other hand, in the Gaussian multiple-access channel with feedback, the cross-term between the user signals seems to lead to creation of energy in the channel, which in turn has a positive effect on the capacity [11]. This difficulty does not arise in the additive energy models, where strict conservation of energy is enforced. As a consequence, feedback does not increase the capacity of the additive energy channel beyond that of the equivalent AWGN without feedback.

As found by Verdú, the capacity region of the AEN channel coincides with that of the equivalent Gaussian channel, at identical noise and energy levels. We have the following chain of inclusions

$$R_{\text{AE-Q}} \subseteq R_{\text{AEN}} = R_{\text{AWGN}} \subseteq R_{\text{fb}}.$$

An open, yet natural, question is the determination of the extent to which the two energy models might represent practical information rates of the physical channel. Since neither phase noise nor the cost of carrier synchronization are included in the AWGN model, an intriguing thought is whether the AE-Q channel may implicitly account for them.

References