AN AXIOMATIC APPROACH TO IMAGE INTERPOLATION

Vincent Caselles†, Jean-Michel Morel† and Catalina Sbert†

† Department of Mathematics, UIB, 07071 Palma de Mallorca, Spain
‡ CEREMADE, University of Paris-Dauphine, 75775 Paris Cedex 16, France

ABSTRACT

We discuss possible algorithms for interpolating data given in a set of curves and/or points in the plane. We propose a set of basic assumptions to be satisfied by the interpolation algorithms which lead to a set of models in terms of possibly degenerate elliptic partial differential equations. The Absolute Minimal Lipschitz Extension model (AMILLE) is singled out and studied in more detail. We show experiments suggesting a possible application, the restoration of images with poor dynamic range.

1. INTRODUCTION

Our purpose in this paper will be to discuss possible algorithms for interpolating scalar data given on a set of points and/or curves in the plane. A number of different approaches using interpolation techniques have been proposed in the literature for "perceptually motivated" coding applications [3]. The underlying image model is based on the concept of "raw primal sketch" [7]. The image is assumed to be made mainly of areas of constant or smoothly changing intensity separated by discontinuities represented by strong edges. The coded information also known as sketch data consists of the geometric structure of the discontinuities and the amplitudes at the edge pixels. In very low bit rate applications the decoder has to reconstruct the smooth areas in between by using the edge information. This can be posed as a scattered data interpolation problem from an arbitrary initial set (the sketch data) under certain smoothness constraints. In the following we assume that a set of curves and points is given and we want to construct a function interpolating these data. Several interpolation techniques using implicitly or explicitly the solution of a partial differential equation have been used in the engineering literature [2]. In the spirit of [1] our approach to the problem will be based on a set of formal requirements that any interpolation operator in the plane should satisfy. Then we show that any operator which interpolates continuous data given on

a set of curves can be given as the viscosity solution of a degenerate elliptic partial differential equation of a certain type. The examples include the Laplacian operator and the minimal Lipschitz extension operator [5] which is related to the work of J. Casas [3].

Let us start with simple heuristic arguments. The main differential operators we discuss here arise immediately from the mere consideration of which kind (linear or nonlinear?) of mean value property an interpolant function \( u(x) \) must have. Let us assume that a function \( u \) is an interpolant of itself. That is, it satisfies for all \( x_0 \in \Omega \) \( u(x_0) = (\text{mean value of } u(x)) \) on a neighbourhood \( \Omega \) no matter what we mean by "mean value". Essentially we have three possibilities: 1) \( u(x_0) \) is a mean value of the neighbouring pixels. Then \( u(x_0) = \frac{1}{4} (u(x_0 + (h, 0)) + u(x_0 - (h, 0)) + u(x_0 + (0, h)) + u(x_0 - (0, h))) \). Taking the difference and letting \( h \to 0 \) it is easily seen by Taylor expansion that this implies

\[
\Delta u(x_0) = \frac{\partial^2 u}{\partial x_1^2}(x_0) + \frac{\partial^2 u}{\partial x_2^2}(x_0) = 0. \tag{1}
\]

2) \( u(x_0) = \text{median value} \{ u(y), \ y \in \Omega(x_0, h) \} \) where \( D(x_0, h) \) is a disk with radius \( h \). In this case \([4]\) letting \( h \to 0 \) we get

\[
\text{curv}(u)(x_0) = \frac{1}{|D_u|^2} D^2 u(D_u^2, D_u^2) = 0. \tag{2}
\]

where \( \text{curv}(u)(x_0) \) is the curvature of the level line passing by \( x_0 \) and \( D_u^2 \) is orthogonal to \( D_u D_u = (u_x, u_y) \) being the gradient of \( u \) and \( D^2 u \) the matrix of the second derivatives of \( u \). (We use the notation \( A(x, y) = \sum_{i,j=1}^2 a_{ij} x_i y_j \) where \( A = (a_{ij})_{i,j=1}^2 \) is a \( 2 \times 2 \) matrix and \( x, y \in \mathbb{R}^2 \)).

3) \( u(x_0) \) is obtained by propagation from neighbouring pixels \([3]\). It is easily seen that if \( u \) is \( C^2 \) at \( x \) and if \( u(x) \) is obtained by this interpolation algorithm then we can write \( u(x) = \frac{1}{h^2} (u(x + h D u) + u(x - h D u) + o(h^2)) \). Letting \( h \to 0 \) and using here again a Taylor expansion \( \text{one gets easily} \)

\[
D^2 u(D_u, D_u) = 0. \tag{3}
\]
This method is inspired from Casas [3] but it must be made clear that the Casas algorithm does not create necessarily a continuous interpolant in contrast with Jensen’s method ([4]).

Our plan is as follows. In Sect. 2 we introduce a formal set of axioms which should be satisfied by any interpolation operator in the plane and derive the associated partial differential equation. In Sect. 3 we discuss several examples of interpolation operators relating them to the set of axioms studied in the previous section. Section 4 is devoted to the numerical analysis of the AMLE model and the display of some experimental results obtained with it.

2. AXIOMATIC ANALYSIS OF INTERPOLATION OPERATORS

Let \( \mathcal{C} \) be the set of continuous simple Jordan curves in \( \mathbb{R}^2 \). For each \( \Gamma \in \mathcal{C} \) let \( \mathcal{F}(\Gamma) \) be the set of continuous functions defined on \( \Gamma \). We shall consider an interpolation operator as a transformation \( E \) which associates with each \( \Gamma \in \mathcal{C} \) and each \( \varphi \in \mathcal{F}(\Gamma) \) a unique function \( E(\varphi, \Gamma) \) defined in the region \( D(\Gamma) \) inside \( \Gamma \) satisfying the following axioms:

(A1) Comparison principle: \( E(\varphi, \Gamma) \leq E(\psi, \Gamma) \) for any \( \Gamma \in \mathcal{C} \) and any \( \varphi, \psi \in \mathcal{F}(\Gamma) \) with \( \varphi \leq \psi \).

(A2) Stability principle: \( E(E(\varphi, \Gamma), \Gamma') = E(\varphi, \Gamma) |_{D(\Gamma')} \) for any \( \Gamma \in \mathcal{C} \) any \( \varphi \in \mathcal{F}(\Gamma) \) and \( \Gamma' \in \mathcal{C} \) such that \( D(\Gamma') \subseteq D(\Gamma) \).

(A3) Regularity principle: Let \( SM(2) \) be the set of symmetric two-dimensional matrices. Let \( A \in SM(2) \) \( p \in \mathbb{R}^2 - \{0\} \Gamma c \in \mathbb{R} \) and \( Q(y) = \frac{A(y-x,y-x)}{2} \) < \( p, y - x > + c \). (where \( < x, y > = \sum_{i=1}^{2} x_i y_i \)). Let \( D(x, r) = \{ y \in \mathbb{R}^2 : \| y - x \| \leq r \} \) and \( \partial D(x, r) \) its boundary. Then

\[
E(Q |_{\partial D(x, r)}, \partial D(x, r)) - Q(x) \quad \frac{r^2}{2} \to F(A, p, c, x) \quad (4)
\]
as \( r \to 0 \) where \( F : SM(2) \times \mathbb{R}^2 - \{0\} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) is a continuous function.

(A4) Translation invariance: \( E(\tau_h \varphi, \Gamma - h) = \tau_h E(\varphi, \Gamma) \Gamma \) where \( \tau_h \varphi(x) = \varphi(x + h) \) \( h \in \mathbb{R}^2 \Gamma, \varphi \in \mathcal{F}(\Gamma) \) \( \Gamma \in \mathcal{C} \). The interpolant of a translated image is the translated of the interpolant.

(A5) Rotation invariance: \( E(R\varphi, R\Gamma) = RE(\varphi, \Gamma) \Gamma \) where \( R\varphi(x) = \varphi(R^t x) \) \( R \) being an orthogonal map in \( \mathbb{R}^2 \). \( \varphi \in \mathcal{F}(\Gamma) \) \( \Gamma \in \mathcal{C} \). The interpolant of a rotated image is the rotated of the interpolant.

(A6) Grey scale shift invariance: \( E(\varphi + c, \Gamma) = E(\varphi, \Gamma) + c \Gamma \) for any \( \Gamma \in \mathcal{C} \) any \( \varphi \in \mathcal{F}(\Gamma) \) \( c \in \mathbb{R} \).

(A7) Linear grey scale invariance: \( E(\lambda \varphi, \Gamma) = \lambda E(\varphi, \Gamma) \) for any \( \lambda \geq 0 \).

(A8) Zoom invariance: \( E(\delta x \varphi, \lambda^{-1} \Gamma) = \delta \lambda E(\varphi, \Gamma) \Gamma \) where \( \delta \lambda \varphi(x) = \varphi(\lambda x) \Gamma \lambda > 0 \). The interpolant of a zoomed image is the zoomed interpolant.

Axioms (A1)–(A3) and (A4) to (A8) are obvious adaptations from the axiomatic developed in [1]. Let us write \( G(A) = F(A, e_1) \Gamma A \in SM(2) \) \( e_1 = (1, 0) \). Then \( G \) is a continuous function of \( A \). Given a matrix

\[
A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},
\]
let us write for simplicity \( G(a, b, c) \) instead of \( G(A) \). Let us also write \( \nu = \frac{Dx}{D_{u_x}} \nu^\perp \) being the unit vector perpendicular to \( \nu \).

Theorem 1 ([4]) Assume that \( E \) is an interpolation operator satisfying (A1) – (A8). Let \( \varphi \in \mathcal{F}(\Gamma) \), \( u = E(\varphi, \Gamma) \). Then \( u \) is a viscosity solution of

\[
G(D^2u(\nu, \nu), D^2u(\nu, \nu^\perp), D^2u(\nu^\perp, \nu^\perp)) = 0 \quad (5)
\]
in \( D(\Gamma) \) with \( u_{\mid \Gamma} = \varphi \). In addition, \( G(A) \) is a non-decreasing function of \( A \) satisfying \( G(\lambda A) = \lambda G(A) \) for all \( \lambda \in \mathbb{R} \). Moreover, if \( G \) is differentiable at 0 then \( G \) may be written as \( G(A) = Tr(BA) \) where \( B \) is a nonnegative matrix, i.e., the equation (5) may be written as

\[
aD^2u(\nu, \nu) + 2bD^2u(\nu, \nu^\perp) + cD^2u(\nu^\perp, \nu^\perp) = 0 \quad (6)
\]
where \( a, c \geq 0, ac - b^2 \geq 0 \).

A further discussion proves that if we require to the interpolation operators described by a smooth function \( G \) to be able to interpolate data given on curves and/or points we are forced to assume model (3) (see [4]).

3. EXAMPLES

Example 1. Given \( \Gamma \in \mathcal{C} \) and \( \varphi \in \mathcal{F}(\Gamma) \) we consider \( E_1(\varphi, \Gamma) \) to be the solution of

\[
\Delta u = 0 \quad \text{in} \ D(\Gamma) \\
u_{\mid \Gamma} = \varphi. \quad (7)
\]
The operator \( E_1 \) satisfies all axioms (A1)–(A8) above. We recall that this operator does not permit to interpolate points. A more general situation is given by the so called p-Laplacian operator (see [4]).

Example 2. Given a domain \( \Omega \) with \( \partial \Omega \in \mathcal{C} \) and \( \varphi \in \mathcal{F}(\partial \Omega) \) we consider \( E_2(\varphi, \partial \Omega) \) to be the viscosity
solution of
\[
D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) = 0 \quad \text{in } \Omega \tag{8}
\]
\[u|_{\partial \Omega} = \varphi.\]

Equation (8) has already been studied in [5] where existence and uniqueness of viscosity solutions was proved. Let us state R. Jensen’s existence result for (8) in a way that makes explicit the fact that we are able to interpolate a datum which is given on a set of curves and points. Let us consider a domain \(\Omega\) whose boundary \(\partial \Omega = \partial_1 \Omega \cup \partial_2 \Omega \cup \partial_3 \Omega\) where \(\partial_3 \Omega\) is a finite union of rectifiable simple Jordan curves \(\Gamma \partial_2 \Omega = \bigcup_{i=1}^{m} C_i\) where \(C_i\) are rectifiable curves homeomorphic to a closed interval and \(\partial_3 \Omega = \{x_i : i = 1, \ldots, N\}\) is a finite number of points. The boundary data to be interpolated is given by a Lipschitz function \(\varphi_1\) on \(\partial_2 \Omega\) two Lipschitz functions \(\varphi_{2,+}, \varphi_{2,-}\) on each curve \(C_i \Gamma\) which coincide on the extreme points of \(C_i \Gamma\) \(i = 1, \ldots, m\) and a constant value \(u_i\) on each point \(x_i \Gamma \) \(i = 1, \ldots, N\). We shall denote by \(C_i^{+} \Gamma C_i^{-}\) the same curve \(C_i\) where we take into account the direction of the normal \(\nu_i^{+}\) \(i \Gamma \nu_i^{-}(x) = -\nu_i^{+}(x) \Gamma x \in C_i\). When we write \(u_{i+} = \varphi_{2,+}\) as in the next theorem we mean that \(u(y) \rightarrow \varphi_{2,+}(x)\) as \(y \rightarrow x\) if \(< y, \nu_i^{+}(x) > < 0 \Gamma x \in C_i\) and similarly for \(u_{i-} = \varphi_{2,-}\).

**Theorem 2** ([4]). Given \(\Omega, \varphi_1, \varphi_{2,+}, \varphi_{2,-}, u_j, i = 1, \ldots, m, j = 1, \ldots, N\), as above then there exists a unique Lipschitz viscosity solution \(u\) of equation (8) in \(\Omega\) with boundary conditions \(u|_{\partial \Omega_1} = \varphi_1, u_{i+} = \varphi_{2,+}, u_{i-} = \varphi_{2,-}\), \(u(x_j) = u_j, i = 1, \ldots, m, j = 1, \ldots, N\). For boundary data \(\varphi\) as above, let \(E_2(\varphi, \partial \Omega)\) be the viscosity solution of (8). Then \(E_2\) satisfies axioms (A1) – (A8).

**Example 3.** The equation (2) gives no interpolation operator since: a) there is no uniqueness of viscosity solutions of (2) \(\Gamma\) there are no viscosity solutions of (2) for general smooth curves \(\partial \Omega\) and boundary data \(\varphi \in F(\partial \Omega)\) ([4]).

**Example 4.** Consider a set of points \(\{x_i : i = 1, \ldots, N\}\) in \(\mathbb{R}^2\). Shepard [8] proposed the following formula
\[
\begin{align*}
f(x) &= \sum_{i=1}^{N} f_i \left| \frac{x - x_i}{|x - x_i|} \right|^{-2} \quad x \neq x_i, \quad i = 1, \ldots, N \\
f(x_i) &= f_i \end{align*}
\]
\[i, j = 1, \ldots, N\] which we solve using a nonlinear over-relaxation method (NLOM). The asymptotic state gives the desired solution of (8).

Figure 1 shows how we can interpolate an image from the quantized level curves obtaining a better result than the corresponding quantized image. Figures 1a) display the original image \(u\) which takes integer values between 0 and 255. Then we quantize it by giving the grey levels between \(r \leq u < (r + 1)\) the
value $r\Gamma r = 0, ..., M \Gamma M = [255/\delta]$. Figure 1b) displays the result of this operation on Figure 1a) for $\delta = 30$. We define the boundary values on the pixels belonging to the boundaries of the level sets $B$ and the neighbouring pixels belonging to the boundary of the complement $B'$. For each pixel $(i, j)$ we define $m(i, j) = \inf \{ r : u(i, j) \geq r \delta \} \Gamma \max \{ r : u(i, j) \geq r \delta \}$. Then we set $u(i, j) = m(i, j) \delta$ if $(i, j) \in B \Gamma u(i, j) = \max \{ r : u(i, j) \geq r \delta \}$ if $(i, j) \in B'$ and we solve Eq. (12) with these boundary data. The result is displayed in Figure 1c).

Acknowledgments. The first and third authors were also partially supported by DGICYT project G reference P1394-1174. We would like to thank Josep R. Casas for interesting discussions and Pierre-Louis Lions who pointed out to us the work of R. Jensen on the AMLE model.

References


